UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1101**
ASSESSMENT PATTERN	:	**MATH1101B**
MODULE NAME	:	**Analysis 1**
DATE	:	**15-May-12**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2011/12-MATH1101B-001-EXAM-197

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State what it means for a sequence $\langle x_n \rangle$ of real numbers to converge to a real number ℓ.

   (b) Use the definition of convergence (not the combination theorem or any other theorems) to prove that
   $$\lim_{n\to\infty} \frac{n^3+1}{4n^3+3} = \frac{1}{4}.$$

   (c) Prove that every convergent sequence of real numbers is bounded.

   (d) Give an example of a bounded sequence that does not converge. Justify your answer.

2. (a) Prove that if $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences of real numbers such that $|b_n| \leq a_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges absolutely.

   (b) Determine (with explanations) whether each of the following series converges or diverges.
   $$\text{(i)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{2}{n^4+1},$$
   $$\text{(iii)} \sum_{n=1}^{\infty} \left(2 - \frac{1}{n^2}\right), \qquad \text{(iv)} \sum_{n=1}^{\infty} \frac{(n!)^2\cos(2n)}{(2n)!}.$$

3. (a) State the definitions of $\lim_{x\to a^+} f(x) = \ell$ and $\lim_{x\to a^-} f(x) = \ell$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
   $$f(x) = \begin{cases} 4x & \text{if } x < 1, \\ 6 - 2x & \text{if } x \geq 1. \end{cases}$$
   Prove carefully, i.e. using ϵ and δ, that $f(x)$ is continuous at $x = 1$.

   (c) State what it means for a sequence of real numbers to be a Cauchy sequence.

   (d) Show that if $\langle x_n \rangle$ is a convergent sequence, then it is a Cauchy sequence.

4. (a) State and prove the sandwich theorem for sequences.
   (b) Using any techniques at your disposal, compute the limit of the sequence
$$x_n = \sqrt{n+1} - \sqrt{n}.$$
   (c) Suppose $S \subset \mathbb{R}$ is a set of real numbers which is nonempty and bounded above. Show that there exists a sequence $\langle x_n \rangle$ with $x_n \in S$ for all n and
$$\lim_{n\to\infty} x_n = \sup S.$$
   (d) Suppose f is a continuous function on a compact interval $[a, b]$. Prove that f attains a maximum value on $[a, b]$.
   Note: You may assume without proof that f is bounded on $[a, b]$.

5. In this question, we consider the function $\exp : \mathbb{R} \to \mathbb{R}$ defined via the power series
$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
This function satisfies the relation $\exp(x + y) = \exp(x) \cdot \exp(y)$; you may assume this without proof.
   (a) Show that $\exp(x) > 0$ for all $x \in \mathbb{R}$.
   (b) Show that $\exp : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function.
   (c) Show that for each $n \in \mathbb{N}$, there exists a constant $C > 0$ such that
$$\exp(x) \geq Cx^n$$
   for all $x \geq 0$.
   (d) Show that for any natural number n, $\lim_{x\to+\infty} x^n \exp(-x) = 0$.

6. (a) State the continuum property of the real numbers.
   (b) Let S denote the set of all *rational* numbers x satisfying $x^2 \leq 3$. What is $\sup S$? Does $\max S$ exist? Explain briefly.
   (c) Suppose $a < b$, $f : [a, b] \to \mathbb{R}$ is a continuous function and λ is a real number with
$$f(a) < \lambda < f(b).$$
   Let $S = \{x \in [a, b] \mid f(x) < \lambda\}$, and suppose $M \in [a, b]$ is an upper bound for S. Show that $f(M) \geq \lambda$.
   (d) Prove that there does not exist any continuous function $f : [-2, 2] \to \mathbb{R}$ satisfying
$$(x^2 - 4)[f(x)]^5 + xf(x) = 1 \quad \text{for all } x \in [-2, 2].$$